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Large relativistic corrections to the positronium decay rate

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Abstract

Relativistic corrections to the positronium decay rate are calculated. They are close to $40(\alpha/\pi)^2$ and $46(\alpha/\pi)^2$ for singlet and triplet states respectively.

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1. The strong disagreement between the experimental value of the orthopositronium decay rate[1]

$$\Gamma_{exp}^{o-Ps} = 7.0482 \pm 16\ \mu s^{-1}. \tag{1}$$

and its theoretical value which includes the order α and $\alpha^2\log(1/\alpha)$ corrections[2, 3, 4, 5]

$$\Gamma_{th}^{o-Ps} = m\alpha^6 \frac{2(\pi^2-9)}{9\pi}\left[1 - 10.28\frac{\alpha}{\pi} - \frac{1}{3}\alpha^2 \log\frac{1}{\alpha}\right] = 7.03830\ \mu s^{-1}. \tag{2}$$

is a serious challenge to modern QED. For the disagreement to be resolved within the QED framework the correction $\sim (\alpha/\pi)^2$, which has not been calculated up to now, should enter the theoretical result (2) with the numerical factor 250 ± 40 which may look unreasonably large.

Such a hope is not as unreasonable however. The argument is as follows[6]. The large, -10.28, factor at the α/π correction to the decay rate (see (2)) means that the typical magnitude of the factor at the α/π correction to the decay amplitude is roughly 5. Correspondingly, this correction squared contributes about $25(\alpha/\pi)^2$ to the decay rate. Indeed, numerical calculations [7] have given factor 28.8 ± 0.2 at $(\alpha/\pi)^2$ in this contribution.

Moreover, it is only natural to expect that the interference of the second-order radiative correction to the amplitude with the zeroth-order amplitude should contribute about twice as much to the decay rate as the square of the first-order correction. In other words, the natural scale for the total second-order radiative correction to the decay rate can be about[8]

$$100(\alpha/\pi)^2. \tag{3}$$

A similar conclusion is made in a recent paper[9] starting from the Pade approximants.

One more class of large contributions to the positronium decay rate is relativistic corrections. A simple argument in their favour is that the corresponding parameter $(v/c)^2 \sim \alpha^2$ is not suppressed, as distinct from that of usual second-order radiative corrections, $(\alpha/\pi)^2$, by the small factor $1/\pi^2 \sim 1/10$. In this article we present the results of calculations of relativistic corrections to the positronium decay rate.

This problem was addressed previously in Refs.[10, 11]. We differ essentially from those authors in the approach to the problem and, which is more essential, in the conclusions made. The origin of the disagreements will be elucidated below.

As to the relativistic correction to the parapositronium decay rate, also obtained in the paper, its calculation was started by us as a warm-up exercise for the much more complicated orthopositronium problem. However, the correction in the singlet case also turns out large, quite close to the sensitivity of the recent experiment[12].

2. The central point when treating the relativistic corrections to the positronium decay rate is as follows. Calculating the decay amplitude we have to integrate the annihilation kernel over the distribution of the electron and positron three-momenta $\vec{p}$. To lowest approximation in v/c the kernel, both for para- and orthopositronium, is independent of those momenta and we are left with the integral over $\vec{p}$ of the nonrelativistic wave function in the momentum representation which is equivalent to $\psi(r=0)$ in the coordinate one.

However, already to first order in $(p/m)^2$ the momentum integral

$$\int d\vec{p}\,(p/m)^2\psi(\vec{p}) \;=\; \int d\vec{p}\,(p/m)^2\frac{8\sqrt{\pi a^3}}{(p^2a^2+1)^2} \tag{4}$$

linearly diverges at $p\to\infty$ ($a = 2/m\alpha$ is the positronium Bohr radius). Crucial for the problem is the following observation. The true relativistic expression for the annihilation kernel does not grow up at $p\to\infty$, as distinct from its expansion in p/m. So, its integral with $\psi(p)$ in fact converges. Let us transform therefore the integral over $|\vec{p}|$ into that from $-\infty$ to $+\infty$ and shift the integration contour into the upper halfplane. We will first come across the wave function pole at $p = im\alpha/2$ and then the relativistic branching point at $p = im$ which originates from the amplitude and is not related by itself to the wave function. It is obviously just the pole contribution which corresponds to the relativistic correction we are looking for. This contribution can be easily calculated and constitutes

$$-\frac{3}{4}\alpha^2\psi(r=0)$$

. In other words, the recipe for treating the relativistic corrections originating from the amplitude is just to make the substitution

$$(p/m)^2 = v^2 \to -\frac{3}{4}\alpha^2 \tag{5}$$

in them. One may wonder about the sign in rhs of this relation. Let us have in mind however that the main contribution to the integral comes from the relativistic cut. That contribution corresponds to usual radiative corrections $\sim\alpha/\pi$ and is of course much larger than the effect $\sim\alpha^2$ we are interested in.

The next point essential for our consideration is the use of noncovariant perturbation theory (see, e.g., Ref.[13]) which allows us to treat in a natural way the positronium binding energy.

Let us start with a more simple case of parapositronium. Here the noncovariant annihilation amplitude can be written as

$$M = 4\pi\alpha V^+(\vec{e}_2\vec{\alpha})\frac{\Lambda_+(\vec{p}-\vec{k}_1)-\Lambda_-(\vec{p}-\vec{k}_1)}{E-\omega-\epsilon(\vec{p}-\vec{k}_1)-\epsilon(p)}(\vec{e}_1\vec{\alpha})U + (1\leftrightarrow 2)\,,$$

$$V = \sqrt{\frac{\epsilon(p)+m}{2\epsilon(p)}}\left(1-\frac{\vec{\alpha}\vec{p}}{\epsilon(p)+m}\right)\begin{pmatrix}0\\ \chi\end{pmatrix},\; U = \sqrt{\frac{\epsilon(p)+m}{2\epsilon(p)}}\left(1+\frac{\vec{\alpha}\vec{p}}{\epsilon(p)+m}\right)\begin{pmatrix}\phi\\ 0\end{pmatrix}. \tag{6}$$

In this expression χ and ϕ are nonrelativistic spinors; $E = 2m - m\alpha^2/4$ is the positronium total energy; $\vec{e}_{1,2}$ and $\vec{k}_{1,2}$ are the polarizations and momenta of the photons; $\omega_1 = \omega_2 = \omega = E/2$ are their frequencies; $\epsilon(p) = \sqrt{m^2+p^2}$;

$$\Lambda_\pm(\vec{p}) = \frac{1}{2}\left(1\pm\frac{\vec{\alpha}\vec{p}+\beta m}{\epsilon(p)}\right)$$

are the projectors onto the positive and negative energy states of a fermion with a momentum $\vec{p}$ correspondingly. The Coulomb interaction in the intermediate state can be neglected since the momentum of one particle in it is close to m.

The expansion of the amplitude in p/m is straightforward. Averaging over the directions of $\vec{p}$ (an S-state is under discussion) and using relation (5) we obtain

$$M + \delta M \; = \; [1 + \alpha^2(\frac{1}{2} + \frac{\sqrt{2}}{8})]M \tag{7}$$

where M is the lowest order annihilation amplitude. The corresponding relative correction to the decay rate is

$$\frac{\delta\Gamma}{\Gamma} = \alpha^2(1 + \frac{\sqrt{2}}{4}) = 1.35\alpha^2. \tag{8}$$

3. The calculation of relativistic corrections for the triplet state decaying into three photons is much more tedious problem. We believe that have managed to simplify it considerably, but still it is too lengthy to be presented in detail in a short letter. So, only its brief outline is given below.

The construction of noncovariant perturbation theory amplitude is straightforward. But it is convenient to treat it in a slightly different way than it was done for the singlet state. First of all rewrite the initial energy E in the perturbative denominators as

$$E = E - 2\epsilon(p) + 2\epsilon(p)$$

and expand the amplitude in

$$E - 2\epsilon(p) = \frac{m\alpha^2}{2}$$

(here we use recipe (5) for p^2/m^2).

Zeroth term of the expansion transforms into usual covariant Feynman amplitude for electron and positron with 4-momenta $(\epsilon(p), \pm\vec{p})$. Now we expand this "covariant" amplitude in p/m, average the terms of second order in p/m over the directions of $\vec{p}$ and make the substitution (5). The interference of this α^2-correction with the lowest order amplitude after the summation over the photons polarizations and integration over the final phase space generates the following correction to the decay rate:

$$\frac{\delta\Gamma_c}{\Gamma} = \alpha^2 \frac{31\pi^2 - 240}{16(\pi^2 - 9)}. \tag{9}$$

This correction is conveniently combined with that originating from the phase space correction. It can be easily demonstrated that the shift of the total energy from $2m$ to $E = 2m - m\alpha^2/4$ changes the phase space and therefore the decay rate by

$$\frac{\delta\Gamma_p}{\Gamma} = -\alpha^2 \frac{1}{4}. \tag{10}$$

In this way we come to the following total "covariant" correction to the decay rate

$$\frac{\delta\Gamma_c + \delta\Gamma_p}{\Gamma} = \alpha^2 \frac{27\pi^2 - 204}{16(\pi^2 - 9)}. \tag{11}$$

We have checked that if one goes over from α^2 to v^2 in Eq.(9) according to prescription (5) and changes the phase space shifting $2m \to 2m + mv^2$, the result coincides with the v^2 correction to the probability of 3γ annihilation of free electron and positron in ${}^3S_1$ state calculated in Ref.[10] (see also Ref.[11]).

The correction to the decay rate induced by the term of first order in $E - 2\epsilon(p) = m\alpha^2/2$ in the expansion of exact amplitude demands numerical calculations which give

$$\frac{\delta\Gamma_n}{\Gamma} = 0.807\alpha^2. \tag{12}$$

Let us note here that the weird term with $\sqrt{2}$ in the correction to the singlet decay rate (see Eqs. (7), (8)) is of the same "noncovariant" origin.

Going back to the triplet decay rate, we have to note that our results for the α^2 corrections themselves differ completely from those of Refs.[10, 11]. It is not so much due to the "noncovariant" correction (12) completely lost there, this correction is not so large numerically. The main problem is that of translating v^2 into α^2. In Ref.[10] they use the prescription $v^2 \to \alpha^2$ which leads to a wrong sign of the α^2 correction (though to a reasonable absolute value). On the other hand, the prescription $v^2 \to -\alpha^2/4$ used in fact in Ref.[11] gives a correct sign, but strongly underestimate the effect. Quite possibly however, there is no direct contradiction between their result and ours, since as it is stated explicitly in Ref.[11], their consideration refers to a part of relativistic corrections only.

4. Let us consider at last the effects originating from relativistic corrections to the wave function $\psi(\vec{r})$ itself. We will use here the Breit equation following to some extent Ref.[6]. Para- and orthopositronium can be treated here in parallel.

The part of the Breit Hamiltonian (BH) that corresponds to the relativistic corrections to the dispersion law of the particles and to their Coulomb interaction,

$$V_c = -\frac{p^4}{4m^3} + \frac{\pi\alpha}{m^2}\delta(\vec{r})\,, \tag{13}$$

can be easily transformed to

$$V_c = \frac{\alpha^3}{8r}. \tag{14}$$

Here and below we omit constant terms in the perturbations (obviously, they do not change the wave function) and substitute $-m\alpha/2$ for ∂_r acting on the ground state positronium wave function.

The next spin-independent term in the BH

$$V_m = -\frac{\alpha}{2m^2 r}\left(p^2 + \frac{1}{r^2}\vec{r}(\vec{r}\vec{p})\vec{p}\right)\,, \tag{15}$$

describes the magnetic electron-positron interaction due to the orbital motion. For the ground state it transforms into

$$V_m = \frac{\alpha^3}{4r} - \frac{\alpha^2}{2mr^2}\,. \tag{16}$$

The last term in BH of interest for our problem is the contact spin-spin interaction

$$V_{ss} = \frac{\pi\alpha}{m^2} A\delta(\vec{r})\,; \quad A = \frac{7}{3}S(S+1) - 2. \tag{17}$$

It is conveniently rewritten as

$$V_{ss} = A\frac{1}{4m}\left[H, \frac{\alpha}{r}\right] + A\frac{\alpha^2}{4mr^2}; \quad H = p^2/m - \alpha/r. \tag{18}$$

The terms $\alpha^3/8r$, $\alpha^3/4r$ from Eqs. (14), (16) taken together shift obviously the coupling constant $\alpha \to \alpha(1 - 3\alpha^2/8)$ which leads to the following relative correction both to the $|\psi(0)|^2$ and decay rate

$$\frac{\delta\Gamma_1}{\Gamma} = -\frac{9\alpha^2}{8}. \tag{19}$$

As easily one can calculate the relative correction due to the commutator term in Eq. (18):

$$\frac{\delta\Gamma_2}{\Gamma} = A\frac{\alpha^2}{2}. \tag{20}$$

Let us turn now to the singular part of the Breit perturbation

$$V_2 = \frac{\lambda}{mr^2}; \quad \lambda = \alpha^2(\frac{A}{4} - \frac{1}{2}). \tag{21}$$

The normalized solution of the radial wave equation

$$\left(\frac{1}{r}\frac{d^2}{r^2}r - \frac{\lambda}{r^2} + \frac{m\alpha}{r} + m\tilde{E}\right) R = 0 \tag{22}$$

is

$$R = 2(m\alpha/2)^{3/2}[1 - \lambda(3 - C)](m\alpha r)^{\lambda} \exp\left[-(1-\lambda)m\alpha r/2\right] \tag{23}$$

where $C = 0.577$ is the Euler constant. The eigenvalue $\tilde{E}$ deviation from $-m\alpha^2/4$ is by itself irrelevant to our problem. The corresponding relative correction to the $|\psi(0)|^2$ and decay rate constitutes obviously

$$\frac{\delta\Gamma_3}{\Gamma} = -2\lambda[(3 - C) - \log(m\alpha r_0)] \tag{24}$$

where $r_0 \sim 1/m$ is the distance at which the annihilation takes place. The logarithmically enhanced part of this correction

$$\alpha^2 \log(1/\alpha) \left\{ \begin{array}{cc} 2, & S = 0 \\ -1/3, & S = 1 \end{array} \right. . \tag{25}$$

has been calculated previously for triplet (see formula (2) and singlet cases in Refs.[4] and [6] respectively. So, we omit it and in this way come to the following total relativistic correction due to the ψ-function modification:

$$\frac{\delta\Gamma_\psi}{\Gamma} = \alpha^2 \left\{ \begin{array}{cc} 31/8 - 2C - 2\log(mr_0), & S = 0 \\ -19/24 + 1/3C + 1/3\log(mr_0), & S = 1 \end{array} \right. . \tag{26}$$

We believe that ± 1 is a fair estimate for the scatter of possible values of $\log(mr_0)$ introduced by the uncertainty in the short-distance cut-off r_0. On the other hand, the cut-off of the logarithmic contribution at the atomic distances has been taken care exactly in our consideration.

Our result for the atomic relativistic correction in orthopositronium, $(-19/24+1/3C)\alpha^2 = -0.6\alpha^2$, differs from that given in Ref.[11]. We cannot explain the disagreement, since the authors of Ref.[11] present only their numerical result for this correction: $1.16\alpha^2$.

5. To summarize, the total relativistic corrections in para- and orthopositronium constitute, respectively,

$$\frac{\delta\Gamma}{\Gamma} = 4.1\alpha^2 = 40(\frac{\alpha}{\pi})^2, \quad S = 0; \tag{27}$$

$$\frac{\delta\Gamma}{\Gamma} = 4.7\alpha^2 = 46(\frac{\alpha}{\pi})^2, \quad S = 1 \tag{28}$$

(it is instructive perhaps to present these relativistic corrections in usual "radiative" units (α/π) as well). The terms with $\log(mr_0)$, omitted here, introduce the errors which we estimate as

$$\pm\, 2\alpha^2 = \pm 20(\frac{\alpha}{\pi})^2, \quad S = 0; \tag{29}$$

$$\pm\, \frac{1}{3}\alpha^2 = \pm 3(\frac{\alpha}{\pi})^2, \quad S = 1. \tag{30}$$

As to the orthopositronium decay rate, our correction (28) and that of Ref.[7], taken together, reduce essentially the gap between the theory and experiment, from $(250\pm 40)(\alpha/\pi)^2$ to $(175 \pm 40)(\alpha/\pi)^2$.

In parapositronium the magnitude of the calculated correction is close to the present experimental accuracy[12]. Here there are no special reasons to expect that true radiative corrections are as large. So the measurement of the effect looks sufficiently realistic.

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References

[1] J.S. Nico, D.W. Gidley, A. Rich and P.W. Zitzewitz, Phys.Rev.Lett. **65**, 1344 (1990).

[2] A. Ore and J.L. Powell, Phys. Rev. **75**, 1696 (1949).

[3] W.E. Caswell, G.P. Lepage, J.R. Sapirstein, Phys. Rev. Lett. **38**, 488 (1977).

[4] W.E. Caswell, G.P. Lepage, Phys. Rev. A. **20**, 36 (1979).

[5] G.S. Adkins, Ann. Phys. (N.Y.) **146**, 78 (1983).

[6] I.B. Khriplovich and A.S. Yelkhovsky, Phys.Lett. **B246**, 520 (1990).

[7] A.P. Burichenko, Yad.Fiz. **56** (1993) 123 [Sov.J.Nucl.Phys. **56** (1993)].

[8] I.B. Khriplovich, A.I. Milstein and A.S. Yelkhovsky, Physica Scripta **T46**, 252 (1993).

[9] M.A. Samuel and G. Li, preprint SLAC-PUB-6318 (1993).

[10] E.A. Kuraev, T.V. Kukhto and Z.K. Silagadze, Yad.Fiz. **51** (1990) 1638 [Sov.J.Nucl.Phys. **51** (1990)].

[11] P. Labelle, G.P. Lepage and U. Magnea, preprint CLNS/93/1199 (1993)?

[12] A.H. Al-Ramadhan and D.W. Gidley, Phys.Rev.Lett. **72** (1994) 1632.

[13] I.B. Khriplovich, A.I. Milstein and A.S. Yelkhovsky, Am.J.Phys. **62** (1994) 70.